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LETTER TO THE EDITOR

Evidence for non-universal exponents in bootstrap percolation

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Abstract. Bootstrap percolation (BP) models are systems where sites are initially randomly occupied. Those sites that do not maintain a suitable local environment of at least m occupied sites are successively removed. There has been considerable discussion regarding the nature of the phase transitions in these systems. For sufficiently large m values first-order transitions occur. These have been the focus of intensive activity in the last few years; however, a consensus concerning their nature has now been achieved for many cases. Current open questions concerning these systems now focus on the universality of critical exponents for the lower m values. Our simulations indicate that for $m=3$ on the simple cubic lattice, the exponent β which characterizes the critical behaviour of the percolation probability, is distinct from that of usual random percolation ($m=0$). We quote $p_c = 0.5717 \pm 0.0005$ and $\beta = 0.6 \pm 0.1$ for $m=3$. We argue that the connectivity exponent ν for $m=2$ should be the same as that of usual random percolation and within the limits of our numerical accuracy we observe that this appears to be the case for $m=3$ on the simple cubic lattice. The implications of this surprising dichotomy are considered.

In bootstrap percolation (BP) [1-5] sites are initially occupied randomly (with probability p) and then all sites that do not have at least m neighbours are permanently removed. This culling continues until no more sites can be removed. In BP models p_c is defined to be the concentration below which no infinite cluster is found in the infinite system in the infinite time limit, and in all cases p_c is greater than or equal to the threshold for usual random percolation on the same lattice. BP with $m=0$ is usual percolation. For $m=1$ BP, isolated sites, and for $m=2$ BP, dangling ends of percolation clusters, are removed. In both cases we expect that the percolation threshold, p_c, remains unchanged on all lattices. For $m=1$ critical exponents should be unchanged. For $m>2$ the threshold is above that of usual random percolation and the nature of the transition is strongly lattice dependent. Now that exact solutions [6] and numerical evidence [7] are largely in agreement concerning the description of the first-order transitions that occur for sufficiently large m values, the universality [2, 7] or otherwise of the exponents of the intermediate m values remains the outstanding unsolved aspect of these systems: are they all percolation-like? The nature of the critical exponents for $m=2$ and $m=3$, and the threshold for $m=3$ on the simple cubic lattice are questions that we address in this letter. We shall show below that our answers also shed some light on questions

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of universality in related systems such as rigidity percolation [8–10] and the zero-temperature behaviour of the dilute three-state antiferromagnetic Potts model on the triangular lattice [11] (3-PAFT).

BP models have recently been reviewed [5], and a full tabulation of numerical results of BP systems with second-order transitions is given in table II of [5]. Two analytic arguments concerning critical exponents of $m=2$ BP models can be made, one concerning distinctive values of β, the critical exponent of the percolation probability, $P(p)\sim(p-p_c)^{\beta}$, the second for identical values of ν, the exponent of the correlation length. The first argument comes from the original BP studies, where Chalupa *et al* showed [1] that on the Bethe lattice, $P(p)$ has an exact solution that gives $\beta=1$ for $m=1$ (identical to the random percolation value, denoted by β_0 below), but $\beta_2=2$ for $m=2$. The applicability of this result for finite lattices may be limited, because, as pointed out by van Enter (private communication), it is not surprising that the values are distinct on the Bethe lattice since for $m=2$, the infinite cluster is identical to the backbone of the percolation infinite cluster as there are no loops. For finite lattices, however, the backbone is distinct from $m=2$ clusters, as there are dangling structures; for example on the square lattice a square attached by a linear chain of sites to the backbone does not carry current but is stable to culling under the $m=2$ rules. However, even in finite lattices the backbone provides the key to the problem as it is well known [12] that the backbone has distinctive $\beta=\beta_{bb}$ values, but identical ν values to usual random percolation. Based on this, we argue that for $m=2$, ν should be identical to that of random percolation on all lattices, because random percolation, backbone percolation and $m=2$ BP all have the same threshold, and thinking in terms of measurement techniques for ν (discussed below) in any simulation, identical numerical estimates will be found. Applying similar thoughts to β we see that since $m=2$ BP has a cluster with $P(p)$ larger than that of the backbone and smaller than that of the random model, we expect that $\beta_0\leqslant\beta_2\leqslant\beta_{bb}$.

In the first BP simulation, Kogut and Leath [2] found distinctive central β values in finite dimensions, on some small samples. Their error bars were (justifiably) large, but it is reasonable to conclude that their β estimates for $m=3$ on both triangular (linear size $L=200$) and simple cubic ($L=35$) lattices are distinct from those of usual percolation. RG studies [13] of both ν and β support the idea of different universality classes. A serious attempt to address the universality question, was made [14] by Khan *et al*, via a Monte Carlo renormalization group (MCRG) study of cell sizes of up to $L=400$ on the triangular lattice. They interpret their results to mean that both $m=2$ and $m=3$ BP on this lattice are in the same universality class as usual percolation. We feel that this conclusion is justified as regards their ν estimates, however, their measurements of the magnetic scaling power y_h, from which a β estimate may be drawn via the relation $\beta=\nu(2-y_h)$ depend on the choice of threshold, and for $m=3$ a quite justifiable choice of threshold leads to a β estimate somewhat above that of usual percolation. A resolution of this question from their data is probably dependent on an independent evaluation of p_c for the $m=3$ case on the triangular lattice.

Our new calculations were motivated by a desire to settle the universality question, and by an intuition that because of the analytic arguments and precedents presented above, there could possibly be differing results for universality of ν and of β values. We felt that independent evaluations with larger lattices of β and ν, (apparently not possible with the analysis methods of [13] and [14]) were needed to clarify this point. We chose the simple cubic lattice because the discrepancy between β estimates of [2] for $m=0$, 2 and 3 is much larger there than for the triangular case and because Ray

and Sahimi were carrying out conductivity studies for $m = 3$ on this lattice and required an accurate threshold estimate.

We have studied systems of size L^3, with $10 \leqslant L \leqslant 110$, for usual percolation, and $m = 2$ and 3 BP at a wide range of concentrations, with 10 000 samples for each case for the smallest samples and several hundred for the larger ones. Helical and periodic boundary conditions were used during the culling process and free boundary conditions during the connectivity checks. An HP 9000/855 was used for the largest samples and longest runs and SUN sparc stations and an Intel 860 chip for the smaller samples. For each choice of m and of (L, p) we calculated both $R(L, p)$, the probability of existence of a spanning cluster, and $P(p)$, the probability that a site belongs to the largest cluster.

We have determined thresholds and ν values by a study of the $R(p)$. Defining the concentration p_x^L to be that concentration at which x% of samples percolate, we measured p_{50}^L as well as two measurements of the width, $W(L)$ of $R(p)$, namely $W(L) = p_{90}^L - p_{10}^L$ and $W(L) = p_{80}^L - p_{20}^L$. We make the assumptions [15] that both $W(L)$ and $p_c - p_{50}^L$ scale as $L^{-1/\nu}$. These enable us to obtain p_c from a plot of p_{50}^L as a function of $W(L)$, without an explicit evaluation of ν, and to obtain ν from a plot of $\log_{10} W(L)$ as a function of $\log_{10} L$, independent of threshold choice.

For $m = 0$, 1 and 2 these methods give identical results since $R(p)$ is the same for all three cases. Our finite samples extrapolate nicely to the infinite system value of $p_c = 0.3116$ from [16]. The results from the $R(p)$ measurements for $m = 3$ are given in figures 1 and 2. From figure 1 we deduce that $p_c = 0.5717 \pm 0.0005$. This is somewhat above the value of 0.568 from [2], the latter value being entirely in line with an extrapolation from smaller samples on our graph. There is a strong curvature in this graph, caused by strong corrections to scaling. In figure 2 we observe that the slope of the $W(L)$ measurements gives $\nu = 0.876 \pm 0.010$, which is quite consistent with the usual ν value of random percolation, $\nu = 0.872 \pm 0.070$ from series [17] or $\nu = 0.875 \pm 0.008$ from simulation [18].

We have evaluated β, independently from either ν or p_c, from a plot of $\log_{10} P(p)$ as a function of $\log_{10}(p - p_c)$. The threshold of the finite system used for this evaluation

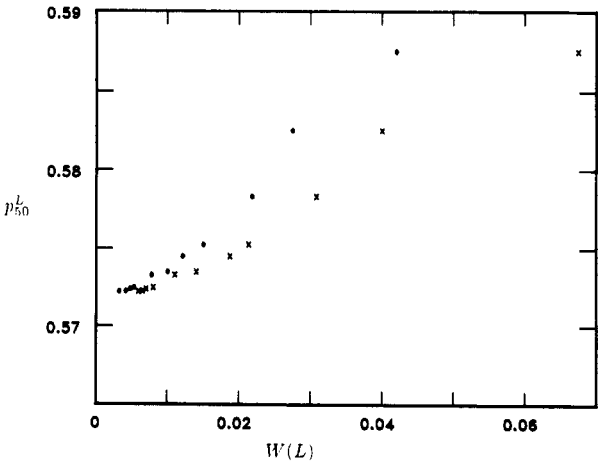

Figure 1. Graphs of p_{50}^L as a function of $W(L)$ for $m = 3$ BP on the simple cubic lattice. The points indicated by ● are measurements of $W(L) = p_{80}^L - p_{20}^L$ and those indicated by × are measurements of $W(L) = p_{90}^L - p_{10}^L$.

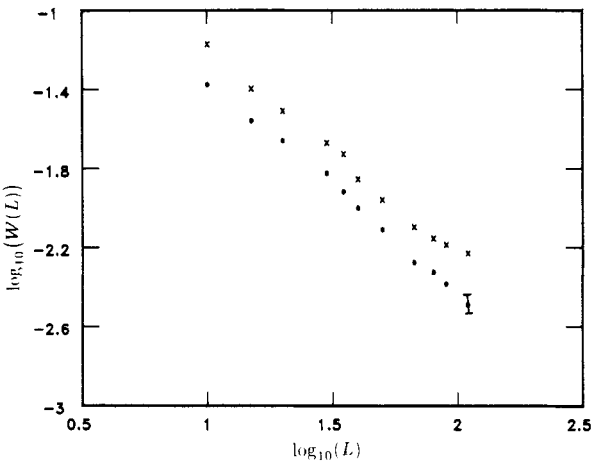

Figure 2. Graphs of $\log_{10} W(L)$ as a function of $\log_{10} L$ for $m = 3$ BP on the simple cubic lattice. The points indicated by ● are measurements of $W(L) = p^L_{80} - p^L_{20}$ and those indicated by × are measurements of $W(L) = p^L_{90} - p^L_{10}$.

is selected to be that which gives the straightest line in this plot. Although this is an obvious way to directly estimate β from a percolation simulation, it is not usually practiced in large simulations, where cluster statistics are not always retained. Therefore we first tested the method with evaluations of β for usual percolation and were pleasantly surprised to find that on a system with $L = 100$ we achieved the straightest line with a slope $\beta_0 = 0.395 \pm 0.010$ at a threshold of 0.313. This slope is in excellent agreement with recent evaluations of $\beta_0 = 0.405 \pm 0.025$ for series results [17] and $\beta_0 = 0.412 \pm 0.010$ from simulations [18]. For smaller samples we found excellent collapse of data at suitable thresholds and similar exponent values.

We present plots of $\log_{10} P(p)$ as a function of $\log_{10}(p - p_c)$ for $L = 80$ systems in figures 3 and 4. In figure 3, the upper line indicates the $m = 0$ data and the lower the

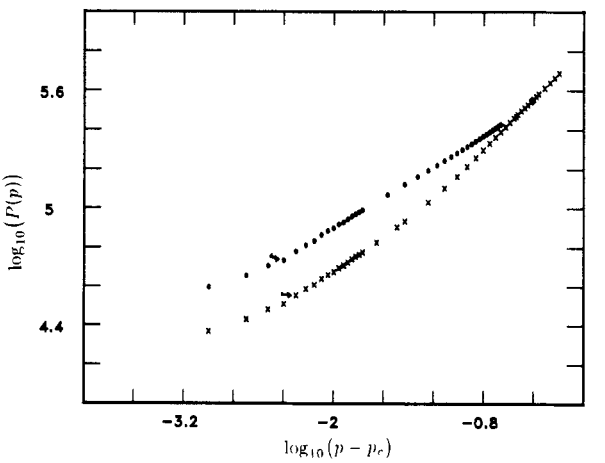

Figure 3. Graphs of $\log_{10} P(p)$ as a function of $\log_{10}(p - p_c)$ for $m = 0$ (indicated by ●) and $m = 2$ (indicated by ×) BP on the simple cubic lattice; $p_c = 0.313$.

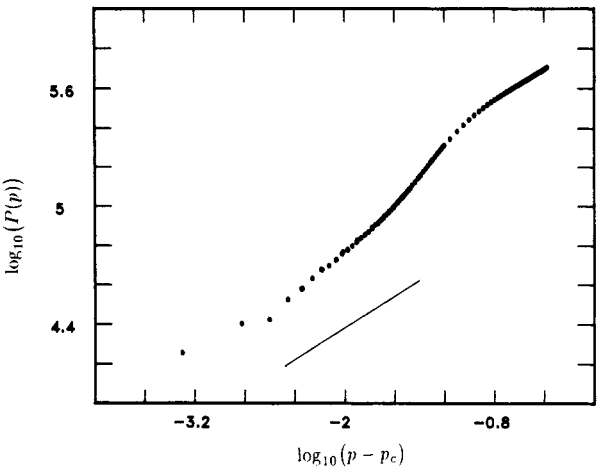

Figure 4. Graphs of $\log_{10} P(p)$ as a function of $\log_{10}(p-p_c)$ for $m=3$ BP on the simple cubic lattice; $p_c=0.5724$. The full line indicates the slope of β_0.

$m=2$. For the highest p values the $m=0$ data falls on top of the $m=2$ data. The arrows on both curves indicate the point where the $m=0$ data begins to scale with $L=100$ ($m=0$) and $L=110$ ($m=2$); points to the left suffer from finite-size distortion. Both plots are drawn at the size-dependent $p_c=0.313$, the results to the right of the arrow are not extremely sensitive to the exact p_c choice. Note that the $m=0$ data follows a single straight line from $-2.4<\log_{10}(p-p_c)<-0.8$, whereas the $m=2$ data follow this line from -2.4 to -1.6 and then crossover to a higher slope. This higher slope has a value of about 0.65, which is quite consistent with the $\beta_2=0.64$ estimate of Kogut and Leath [2]. The two distinct regions of slope of about 0.4 and slope of about 0.65 occur for a wide range of sample sizes and p_c values; comparisons between different sized samples suggest that the region of $m=2$ data which has the same slope as the $m=0$ is not a finite-size effect, but reflects the true critical behaviour of the system. Apparently the critical region is very small and there is a rapid crossover to the region of larger slope. We conclude that $\beta_2=\beta_0$, is consistent with our data.

A very cursory glance at figure 4 ($p_c=0.5724$) for $m=3$ shows that here the shape of the curve is quite distinct from those of $m=0$ and 2. We were unable to select any threshold choice where the s-curvature disappeared, although the slope of the line below $\log_{10}(p-p_c)=-2.2$ did depend on exact threshold choice. Here again very nice scaling of different sized samples is seen for $\log_{10}(p-p_c)>-2.4$ for samples of different sizes. There appears to be a crossover between a slope of 0.60 ± 0.03 and a slope of 0.77 ± 0.10 at about $\log_{10}(p=p_c)=-1.6$. Again the higher slope is consistent with the estimate of Kogut and Leath [2] of 0.82. For ease of comparison, the slope of β_0 is indicated by the full line in figure 4; both data slopes are clearly higher. Above $\log_{10}(p-p_c)=-0.8$ a third slope is seen, indicating a rich behaviour of corrections to scaling. We do note that some region with the slope of 0.6 ± 0.03 is seen in samples as small as $L=35$; it is seen very clearly in the $L=110$ data. (For $L=80$ and p_c of the infinite sample limit (rather than the finite-size p_c of the plotted data), we see that the first slope is rather higher than for thresholds closer to p_{50}^{80}, but there is then a small region of lower slope before the crossover into the higher slope further from the critical region.) We conclude that $\beta_3=0.6\pm0.1$ for $m=3$ BP.

Caution in drawing any conclusion based purely on the fairly small samples of most studies made to date (our own included) is advisable. The simulation analyses explicitly assume, and the renormalization group (RG) calculations implicitly require, the usual kind of second-order finite-size scaling. We have no proof that this is correct, and it appears that corrections to scaling play a larger role here, than they do for the usual percolation cases. Another reason to take evidence of differing exponents with caution, can be motivated from the related rigidity percolation [8] problems, where initial studies suggested distinct exponents from those of random percolation whereas later calculations with larger samples [9] indicated otherwise. Taking into account these warnings, our overall conclusions are that ν is independent of m, and that $\beta_0 = \beta_1 = \beta_2$, but that β_3 is distinct. We find that $p_c = 0.5717 \pm 0.0005$ for $m = 3$ BP. It would be interesting to understand the nature of the crossovers in the $m = 2$ and $m = 3$ models. Ray and Sahimi have recently found that our new $m = 3$ threshold gives a conductivity exponent that is the same as random percolation. This suggests that $m = 3$ BP is of a similar type to backbone percolation, with β being the only exponent that is distinct from usual percolation. We conjecture that rigidity percolation may also resemble backbone percolation in this regard, as there too it is the β values that are so different from the usual case. In contrast, current numerical results [11] give both β and ν of random percolation for percolation of the zero temperature dilute 3-PAFT. If the latter result holds up for larger samples then distinctiveness of β values may be an important qualitative difference between rigidity percolation and the zero temperature dilute 3-PAFT, and a useful way to classify different types of generalized percolation models.

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